UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH1202**

MODULE NAME : **Algebra 2**

DATE : **02-May-07**

TIME : **14:30**

TIME ALLOWED : **2 Hours 0 Minutes**

2006/07-MATH1202A-001-EXAM-205

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*

The use of an electronic calculator is **not** *permitted in this examination.*

1. Let H be a subset of a group G. Give necessary and sufficient conditions for H to be a subgroup of G. In each of the following cases, determine if H is a subgroup of G or not, justifying your answer:

   (i) $G = \mathbb{R} - \{0\}$ under $\times$, $H = \{x \in G : x \geq 1\}$;

   (ii) $G = \mathbb{R} - \{0\}$ under $\times$, $H = \{\frac{1}{2}, 1, 2\}$;

   (iii) $G = \mathbb{R} - \{0\}$ under $\times$, $H = \{2^i : i \in \mathbb{Z}\}$;

   (iv) G is any group, G_1 and G_2 are subgroups of G and $H = G_1 \cap G_2$;

   (v) G is any group, $H = \{z \in G : zg = gz \text{ for all } g \in G\}$.

2. (a) Prove that if G is a finite group and H a subgroup, then $|H|$ divides $|G|$.

   (b) Prove that any subgroup of a cyclic group is cyclic. Find all subgroups of C_{10}, justifying your answer.

3. (a) Let A be an $n \times n$ matrix. Give the definition of $\det(A)$. State, without proof, the effect on the determinant of each type of elementary row operation.

   (b) Evaluate $\det \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 3 & 0 & 2 & 1 \\ 0 & 2 & -1 & 0 \end{pmatrix}$.

   (c) Find $\det \begin{pmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{pmatrix}$, expressing your answer as a product of linear factors.

4. (a) Let A be an $n \times n$ matrix over $\mathbb{R}$. Give the definition of:

(i) an *eigenvalue* λ of A;

(ii) an *eigenvector* $\mathbf{v}$ of A;

(iii) the *characteristic polynomial* $c_A(t)$ of A;

(iv) A is *diagonalizable* (over $\mathbb{R}$).

State a necessary and sufficient criterion, in terms of eigenvectors, for a matrix to be diagonalisable.

(b) Prove that if A has n distinct eigenvalues, then A is diagonalisable.

(c) Give an example of (i) a 2×2 matrix which is not diagonalizable (ii) a 3×3 matrix with two distinct eigenvalues which is diagonalizable, justifying your anwers.

5. Let $A = \begin{pmatrix} 7 & -12 \\ 2 & -3 \end{pmatrix}$.

(i) Find an invertible matrix P such that $P^{-1}AP$ is diagonal.

(ii) Find A^n (for positive integers n).

(iii) Solve the system of differential equations

$$\begin{aligned} x_1' &= 7x_1 - 12x_2 \\ x_2' &= 2x_1 - 3x_2 \end{aligned}$$

given that $x_1(0) = 1$, $x_2(0) = 0$.

6. (a) Define what it means to say that $\langle\, , \,\rangle$ is an *inner product* on a real vector space V. Prove that if $\langle\, , \,\rangle$ is an inner product on V, then, for all $\mathbf{u}, \mathbf{v} \in V$, $\langle \mathbf{u}, \mathbf{v} \rangle \leq |\mathbf{u}|\,|\mathbf{v}|$.

(b) Let $A = \begin{pmatrix} 0 & -2 & -2 \\ -2 & 3 & 4 \\ -2 & 4 & 3 \end{pmatrix}$. Find an orthogonal matrix P such that $P^{-1}AP$ is diagonal.

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